Step 1 Part Two; $\operatorname{Re} s > 0$

Theorem 6.11 *For $s \neq 1$ we have*

$$\sum_{1 \leq n \leq N} \frac{1}{n^s} = 1 + \frac{1}{s-1} + \frac{N^{1-s}}{1-s} - s \int_1^N \{u\} \frac{du}{u^{s+1}} \tag{8}$$

*for **integer** $N \geq 1$.*

Proof Either apply Euler Summation or, as here, Partial Summation from first principles. (Note that this argument, for $s = 1$, has been seen before in the previous Chapter.)

$$\begin{aligned} \sum_{1 \leq n \leq N} \frac{1}{n^s} &= \sum_{1 \leq n \leq N} \left(\frac{1}{N^s} - \left(\frac{1}{N^s} - \frac{1}{n^s} \right) \right) \\ &= \frac{N}{N^s} - \sum_{1 \leq n \leq N} \int_n^N (-s) \frac{du}{u^{s+1}}. \end{aligned}$$

Interchanging summation and integration,

$$\begin{aligned} \sum_{1 \leq n \leq N} \frac{1}{n^s} &= \frac{N}{N^s} + s \int_1^N \sum_{1 \leq n \leq u} 1 \frac{du}{u^{s+1}} \\ &= \frac{N}{N^s} + s \int_1^N [u] \frac{du}{u^{s+1}} \\ &= \frac{N}{N^s} + s \int_1^N u \frac{du}{u^{s+1}} - s \int_1^N (u - [u]) \frac{du}{u^{s+1}} \\ &= \frac{N}{N^s} + \frac{s}{1-s} \left(N^{1-s} - 1 \right) - s \int_1^N \{u\} \frac{du}{u^{s+1}}. \end{aligned} \tag{9}$$

■

Let $N \to \infty$, for which we will require $\operatorname{Re} s > 1$.

Theorem 6.12 *For $\operatorname{Re} s > 1$,*

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{\{u\}}{u^{1+s}} du. \tag{10}$$

Proof Let $N \to \infty$ in (9) , when

$$\left|N^{1-s}\right| = N^{1-\operatorname{Re} s} \to 0$$

since $1 - \operatorname{Re} s < 0$. Also

$$\left|\frac{\{u\}}{u^{1+s}}\right| \leq \frac{1}{u^{1+\operatorname{Re} s}} \leq \frac{1}{u^2} \tag{11}$$

since $\operatorname{Re} s > 1$. Hence, the integral converges and

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{1+s}}du \tag{12}$$

as required. ■

The **Important Observation** to make is that the integral on the right hand side of (10) converges (absolutely) in the *larger* half-plane $\operatorname{Re} s > 0$ for

$$\left|\int_1^\infty \frac{\{u\}}{u^{1+s}}du\right| \leq \int_1^\infty \frac{1}{u^{1+\sigma}}du = \frac{1}{\sigma}.$$

Definition 6.13 *For* $\operatorname{Re} s > 0$ *define the Riemann zeta function by* (10).

The content of Theorem 6.12 is that $\zeta(s)$ defined by (10) for $\operatorname{Re} s > 0$ agrees with the series definition (6) for $\operatorname{Re} s > 1$.

What of $\zeta(s)$ defined by (10) in $\operatorname{Re} s > 0$; apart from the pole at $s = 1$ is it holomorphic in $\operatorname{Re} s > 0$?

Theorem 6.14 $\zeta(s)$ *defined by* (10) *for* $\operatorname{Re} s > 0$ *is holomorphic in that half-plane apart from a simple pole, residue* 1, *at* $s = 1$.

Proof not given. ■

There is a version of Weierstrass's M-test for integrals that shows that the integral in (10) converges uniformly in the half-plane $\operatorname{Re} s \geq \delta$ for any $\delta > 0$.

And then there is a version of Weierstrass's Theorem for *integrals*, see the Background: Complex Analysis II notes, which shows that if an integral of a holomorphic function converges uniformly then it is holomorphic.

Unfortunately Weierstrass's Theorem for integrals result is not directly applicable here since it requires the integrand to be a continuous function yet in this case the integrand in (10) , $\{u\}\, u^{-1-s}$, is **not** a continuous function of u for fixed s. Instead the integral has to be split into a sum of integrals over intervals $(n, n+1)$, $n \geq 1$, and Weierstrass's Theorem for *series* applied. We can then deduce that the integral in (10) is holomorphic in $\operatorname{Re} s > 0$. (See Appendix.) Hence

Example 6.15 *We have that*

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{s-1} \tag{13}$$

is holomorphic (analytic) on $\operatorname{Re} s > 1$ *and*

$$1 - s\int_1^{\infty} \frac{\{u\}}{u^{1+s}} du \tag{14}$$

is a function analytic on $\operatorname{Re} s > 0$ *and which agrees with* (13) *on* $\operatorname{Re} s > 1$.

We have now an example of

Definition 6.16 *Assume that* $F(z)$ *is analytic on domain* $\mathcal{F}$ *and* $G(z)$ *is analytic on* $\mathcal{G}$ *where* $\mathcal{G} \supseteq \mathcal{F}$*. If* $G(z) = F(z)$ *for all* $z \in \mathcal{F}$ *we say that* G *is an* ***analytic continuation*** *of* F *to* $\mathcal{G}$.

Hence (14) is an analytic continuation of (13) to $\operatorname{Re} s > 0$.

On a problem sheet you are asked to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} = \left(1 - \frac{1}{2^{s-1}}\right) \zeta(s),$$

for $\operatorname{Re} s > 1$, and that the left hand-side converges for $\operatorname{Re} s > 0$. It can be shown that the Dirichlet Series on the left converges *uniformly* in $\operatorname{Re} s \geq \delta$ for any $\delta > 0$, and so is holomorphic in $\operatorname{Re} s > 0$. Thus we have another analytic continuation of $\zeta(s)$ to $\operatorname{Re} s > 0$.

Assume that $F(z)$ is analytic on domain $\mathcal{F}$ containing a convergent sequence of points along with the limit point. Further assume that there are two analytic continuations G_1 and G_2 of F to a larger domain $\mathcal{G} \supseteq \mathcal{F}$. Then, since G_1 and G_2 will be equal on this convergent sequence and limit point, Theorem 6.9 implies $G_1(z) = G_2(z)$ on $\mathcal{G}$. That is, the analytic continuation is unique. This means that the word '*an*' in Definition 6.16 can be replaced by '*the*'. And it also means that (10) is the only way of extending $\zeta(s)$ to $\operatorname{Re} s > 0$.